88087205

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MATHEMATICS
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PAPER 3 – SETS, RELATIONS AND GROUPS

Thursday 13 November 2008 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 12]*

A, B, C and D are subsets of $\mathbb{Z}$.

$$A = \{m \mid m \text{ is a prime number less than } 15\}$$

$$B = \{m \mid m^4 = 8m\}$$

$$C = \{m \mid (m+1)(m-2) < 0\}$$

$$D = \{m \mid m^2 < 2m + 4\}$$

(a) List the elements of each of these sets. *[4 marks]*

(b) Determine, giving reasons, which of the following statements are true and which are false.

(i) $n(D) = n(B) + n(B \cup C)$

(ii) $D \setminus B \subset A$

(iii) $B \cap A' = \varnothing$

(iv) $n(B \Delta C) = 2$ *[8 marks]*

2. *[Maximum mark: 10]*

A binary operation is defined on $\{-1, 0, 1\}$ by

$$A \odot B = \begin{cases} -1, & \text{if } |A| < |B| \\ 0, & \text{if } |A| = |B| \\ 1, & \text{if } |A| > |B|. \end{cases}$$

(a) Construct the Cayley table for this operation. *[3 marks]*

(b) Giving reasons, determine whether the operation is

(i) closed;

(ii) commutative;

(iii) associative. *[7 marks]*

3. *[Maximum mark: 10]*

Two functions, F and G, are defined on $A = \mathbb{R} \setminus \{0, 1\}$ by

$$F(x) = \frac{1}{x}, \ G(x) = 1 - x, \text{ for all } x \in A.$$

(a) Show that under the operation of composition of functions each function is its own inverse. *[3 marks]*

(b) F and G together with four other functions form a closed set under the operation of composition of functions.

Find these four functions. *[7 marks]*

4. *[Maximum mark: 13]*

Determine, giving reasons, which of the following sets form groups under the operations given below. Where appropriate you may assume that multiplication is associative.

(a) $\mathbb{Z}$ under subtraction. *[2 marks]*

(b) The set of complex numbers of modulus 1 under multiplication. *[4 marks]*

(c) The set $\{1, 2, 4, 6, 8\}$ under multiplication modulo 10. *[2 marks]*

(d) The set of rational numbers of the form

$$\frac{3m+1}{3n+1}, \text{ where } m, n \in \mathbb{Z}$$

under multiplication. *[5 marks]*

5. *[Maximum mark: 15]*

Three functions mapping $\mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ are defined by

$$f_1(m, n) = m - n + 4; \;\; f_2(m, n) = |m|; \;\; f_3(m, n) = m^2 - n^2.$$

Two functions mapping $\mathbb{Z}\to\mathbb{Z}\times\mathbb{Z}$ are defined by

$$g_1(k) = (2k, k); \;\; g_2(k) = (k, |k|).$$

(a) Find the range of

(i) $f_1 \circ g_1$;

(ii) $f_3 \circ g_2$. *[4 marks]*

(b) Find all the solutions of $f_1 \circ g_2(k) = f_2 \circ g_1(k)$. *[4 marks]*

(c) Find all the solutions of $f_3(m, n) = p$ in each of the cases $p = 1$ and $p = 2$. *[7 marks]*